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Scaling relationship in the log σ – log $\dot{\varepsilon}$ creep and stress–relaxation curves and the plastic equation of state

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The log σ–log $\dot{\epsilon}$ curves, obtained either during creep or stress-relaxation experiments, for different strain levels or initial stresses, frequently show a scaling behaviour. It will be discussed if the scaling relationship is a sufficient condition to ensure the existence of a state variable, dependent on σ and $\dot{\epsilon}$. The results will be applied to some constitutive equations used to describe the plastic behaviour of metals and to creep and stress-relaxation data in Zircaloy-4. Finally, it is concluded that the constitutive equations considered define a state variable in a certain range of σ and $\dot{\epsilon}$, limited by the values of the constants characteristic of the material.

1. Introduction

Hart [1, 2] and Hart *et al.* [3] have proposed a phenomenological theory of plastic deformation based on a plastic equation of state, where each deformation state of the material is a unique state of plastic hardness that can be characterized by a well defined state variable, the hardness. The existence of a plastic equation of state, in terms of stress, plastic strain rate, temperature and hardness, can be tested by performing load relaxation experiments, since stress–plastic strain rate data are obtained at essentially constant hardness. According to this theory, the equation of state can be written as

$$F(\sigma, \dot{\epsilon}, T, \sigma^*) = 0 \qquad (1)$$

where σ is the applied stress, $\dot{\epsilon}$ the strain rate, T the absolute temperature and σ^* the hardness.

Several stress-relaxation log σ–log $\dot{\epsilon}$ curves, obtained in metals and alloys with different crystalline structures, have shown a scaling relationship, i.e. the log σ–log $\dot{\epsilon}$ curves obtained at different hardness, in a given material, can be superposed by translations [3, 4]. In fact, any stress-relaxation curve can be superposed by a translation ($\Delta \log \sigma$, $\Delta \log \dot{\epsilon}$) onto any of the others in such a way that the overlapping segments of each curve match within the experimental error. The translation path, μ, is given by

$$\mu = \Delta \log \sigma / \Delta \log \dot{\epsilon}. \qquad (2)$$

This scaling property has been taken as a proof of the uniqueness of the log σ–log $\dot{\epsilon}$ curves and of the existence of an equation of state for the materials [3].

It is the purpose of this paper to show in what conditions Equation 1, if it can be appropriately transformed and with the restriction imposed by Equation 2, is an equation of state.

Finally, the results will be applied to some constitutive equations, proposed in the literature to describe the plastic behaviour of some materials.

2. Theory

2.1. Scalar field with a scaling behaviour

Equation 1, under isothermal conditions, can be

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written in general as

$$F(x, y, z) = 0 \quad (3)$$

and, if this equation can be written explicitly as

$$y = y(x, z) = g(Ax + Bz) + f(z) \quad (4)$$

where A and B are real constants, g and f are real functions which are continuous, single-valued and differentiable, it is easy to show that

$$y(x + \Delta x, z_2) - y(x, z_1) = f(z_2) - f(z_1) \quad (5)$$

and

$$\partial y/\partial x|_{x, z_1} = \partial y/\partial x|_{x+\Delta x, z_2} \quad (6)$$

where $\Delta x = B(z_1 - z_2)/A$ for all z_1 and z_2. Then, for a given z_1 and z_2, the following functions can be defined in the x,y plane:

$$y(x, z_1) = g(Ax + Bz_1) + f(z_1) \quad (7a)$$

$$y(x, z_2) = g(Ax + Bz_2) + f(z_2) \quad (7b)$$

which are related, point to point, by the translation

$$\Delta y = (A/B)\{[f(z_2) - f(z_1)]/(z_1 - z_2)\}\Delta x. \quad (8)$$

The translation given by Equation 8 depends on the values of z and does not describe the experimental relationship given by Equation 2. If $f(z)$ is a linear function, however, i.e.

$$f(z) = az + b \quad (9)$$

where a and b are real constants, then

$$\Delta y = (-aA/B)\,\Delta x \quad (10)$$

and the translation path is independent of z.

By using Equations 4, 9 and 10 it can be established whether a relationship of the type given by Equation 1 can give, explicitly, a scalar field with a scaling behaviour. If so the relationship allows the slope of the translation path to be determined in terms of the constants.

2.2. The scalar field and the equation of state

The relationship given by Equation 1 is an equation of state if any one of the variables is an unique function of the others or, which is equivalent, if the differential of each variable is a perfect differential [5]. With these conditions, the particular case given by Equation 4, which can be written as

$$F(x, y, z) = y - g(Ax + Bz) - f(z) = 0 \quad (11)$$

will be analysed.

Even if the functions g and f are unknown explicitly, it can be assumed that they define a function $z = \phi(x, y)$ and the differential

$$dz = (\partial\phi/\partial x)_y\, dx + (\partial\phi/\partial y)_x\, dy. \quad (12)$$

The necessary and sufficient condition for this to be a perfect differential is

$$\frac{\partial}{\partial y}(\partial\phi/\partial x)_y = \frac{\partial}{\partial x}(\partial\phi/\partial y)_x \quad (13)$$

and is continuous. By using the change of variables $u = Ax + Bz$ and the theorems for derivatives of implicit functions [5], it can be shown that

$$(\partial\phi/\partial x)_y = -\frac{(\partial F/\partial x)_{y,z}}{(\partial F/\partial z)_{x,y}} = -g'(u)A/[g'(u)B + f'(z)] = -G(x, y)/H(x, y) \quad (14a)$$

and

$$(\partial\phi/\partial y)_x = -\frac{(\partial F/\partial y)_{x,z}}{(\partial F/\partial z)_{x,y}} = 1/[g'(u)B + f'(z)] = -M(x, y)/H(x, y). \quad (14b)$$

Then,

$$\frac{\partial}{\partial y}(\partial\phi/\partial x)_y = \frac{\partial}{\partial x}(\partial\phi/\partial y)_x = -[AB\,g''(u)f'(z) - A\,g'(u)f''(z)]/H^2\,(\partial\phi/\partial y)_x \quad (15)$$

The condition of continuity in Equation 13 implies that $H(x, y) \neq 0$, i.e.

$$g'(u)B + f'(z) \neq 0 \quad (16)$$

and for the particular case of Equation 9

$$g'(u) \neq -a/B \quad (17)$$

Then, a scalar field with a scaling behaviour will be a consequence of an equation of state, unless the condition implied by Equation 17 is not obeyed. If this is the case, the scalar field will be an equation of state only in a restricted domain of the variables, where Equation 17 is satisfied.

3. Applications

The results of the previous section will be applied to some specific constitutive equations, proposed in the literature, for the description of the plastic behaviour of materials. Furthermore, as a check of the validity of the analysis performed, the formalism will be applied to the state equation for ideal gases.

3.1. State equation for ideal gases

The very well known state equation for ideal gases

$$pV = nRT \tag{18}$$

does not generate a scalar field with scaling behaviour. However, if the equation is written as

$$\log p + \log V = \log (nR) + \log T \tag{19}$$

and the variables changed to

$$x = \log p, \qquad y = \log V \qquad \text{and } z = \log T \tag{20}$$

and on assuming, without loss of generality, that the mass is constant, i.e.

$$\log (nR) = \text{constant} = C \tag{21}$$

the result is

$$y(x, z) = (-x + z) + C. \tag{22}$$

Equations 22 is similar to Equation 4 with

$$g(Ax + Bz) = -x + z \tag{23a}$$

and

$$f(z) = C \tag{23b}$$

where $A = -1$, $B = 1$, $a = 0$ and $b = C$. Equation 10 has infinite solutions in this case. This is expected since Equation 22 defines straight lines in the x, y plane, for constant z, which are parallel and admit any translation path. Equation 17 becomes for this case $g'(u) = 1 \neq 0 = -a/B$ for all x and y confirming that no discontinuities are present.

3.2. Hart's phenomenological model

Hart [2] has proposed a deformation model consisting of essentially two parallel branches. At low homologous temperatures, the constant hardness $\log \sigma$–$\log \dot{\epsilon}$ curves are represented by

$$\sigma = \sigma^* + \mathscr{M}(\dot{a}^*)^{1/M}\, \dot{\epsilon}^{1/M} \tag{24}$$

where $\dot{a}^*$ is a rate parameter, $\mathscr{M}$ the anelastic modulus and M is a constant. Rearranging Equation 24 and making the change of variables $x = \log \sigma$, $y = \log \dot{\epsilon}$ and $z = \log \sigma^*$ gives

$$y = Mz - M \log [\mathscr{M}(\dot{a}^*)^{1/M}] + M \log \{\exp [(x - z)/\log e] - 1\} \tag{25}$$

which is an equation similar to Equation 4 with

$$g(Ax + Bz) = M \log \{\exp [(x - z)/\log e] - 1\} \tag{26}$$

where $A = 1, B = -1$ and

$$f(z) = Mz - M \log [\mathscr{M}(\dot{a}^*)^{1/M}] = az + b \tag{27}$$

where $a = M$ and $b = -M \log [\mathscr{M}(\dot{a}^*)^{1/M}]$.

According to Equation 10, the translation path is

$$\Delta y = -(aA/B)\, \Delta x \tag{28a}$$

or

$$\Delta \log \sigma = (1/M)\, \Delta \log \dot{\epsilon} \tag{28b}$$

The condition of continuity, given by Equation 17, in this case leads to

$$\exp [(x - z)/\log e]\, [(1/\log e) - 1] \neq -1 \tag{29}$$

which is always satisfied for all values of x and z. It is easy to show that this means that two $\log \sigma$–$\log \dot{\epsilon}$ curves, for different σ^*, never intersect.

At high homologous temperatures, according to Hart's theory, the constant hardness $\log \sigma$–$\log \dot{\epsilon}$ curves can be represented by the flow law

$$\sigma = \sigma^* \exp [-(\dot{\epsilon}^*/\dot{\epsilon})^{\lambda}] \tag{30}$$

where λ is a constant and $\dot{\epsilon}^*$ a rate parameter. Rearranging Equation 30 and making the change of variables $x = \log \dot{\epsilon}$, $y = \log \sigma$ and $z = \log \sigma^*$ leads to

$$y = z - (\log e) \exp [(\lambda/\log e)(\log \dot{\epsilon}^* - x)]. \tag{31}$$

If Equation 31 is of the form of Equation 4, then

$$\log \dot{\epsilon}^* = m \log \sigma^* + \log K \tag{32}$$

which is the relationship given by Hart *et al.* [3]. With this restriction, Equation 31 can be written as

$$y = z - (\log e) \exp [(\lambda \log K/\log e)] \times \exp [\lambda(mz - x)/\log e] \tag{33}$$

Then (Equations 4 and 9)

$$g(Ax + Bz) = -(\log e) \exp [(\lambda \log K/\log e)] \times \exp [\lambda(mz - x)/\log e] \tag{34}$$

where $f(z) = z$, $A = -1$, $B = m$, $a = 1$ and $b = 0$. The slopes of the translation path, given by Equation 10, is

$$\Delta \log \sigma = (1/m)\, \Delta \log \dot{\epsilon}. \tag{35}$$

Equation 17, applied to this case, predicts dis-

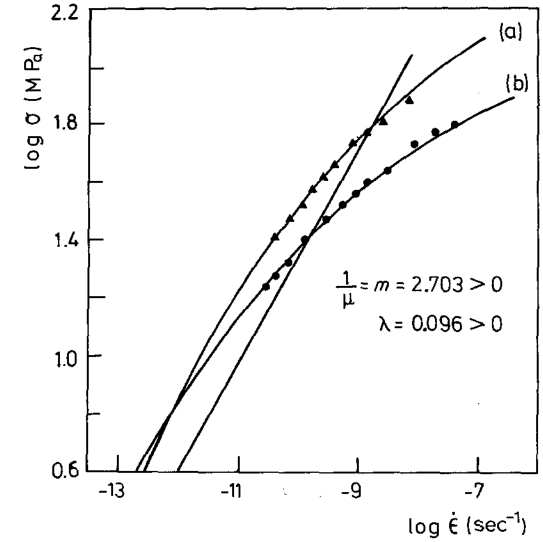

Figure 1 Stress-relaxation data, in bending, for cold-worked Zircaloy-4, taken from Fig. 3b(C) of [4] and fitted to Equation 30. $K = 1.94 \times 10^{-13}$ MPa^{-m} sec^{-1}. The straight line gives the translation path.

continuities since

$$g'(u) = -\lambda \exp[(\lambda \log K/\log e)] \times \exp[\lambda(mz - x)/\log e] \neq -1/m \quad (36)$$

which might not be obeyed, for a certain range of x and z, if $\lambda > 0$ and $m > 0$. It can be easily shown that this discontinuity means that two curves, at constant z (constant σ^*), have a crossing point in the x, y plane given by

$$\exp[-(\lambda x/\log e)] = (z_1 - z_2)/\log e \times \exp(\lambda \log K/\log e)\,[\exp(\lambda m z_1/\log e) - \exp(\lambda m z_2/\log e)] \quad (37)$$

Some experimental creep and stress-relaxation curves for cold-worked Zircaloy-4†, reported by Povolo and associates [4, 6], will be analysed as an example. These data have been interpreted in terms of Equation 30. Fig. 1 shows the crossing of two stress-relaxation curves, measured in bending, obtained from Fig. 3b(C) of [4]. The parameters are: (a) $\sigma^* = 520$ MPa, $\dot{\epsilon}^* = 4.25 \times 10^{-6}$ sec^{-1}; (b) $\sigma^* = 217.8$ MPa, $\dot{\epsilon}^* = 4.05 \times 10^{-7}$ sec^{-1}. Fig. 2 shows the crossing of two $\log \sigma$–$\log \dot{\epsilon}$ creep curves, at two different strain levels, obtained from Fig. 3 of [6]. The parameters are: (a) $\sigma^* = 1847$ MPa, $\dot{\epsilon}^* = 1.76 \times 10^{-3}$ sec^{-1}; (b) $\sigma^* = 1248$ MPa, $\dot{\epsilon}^* = 8.07 \times 10^{-5}$ sec^{-1}.

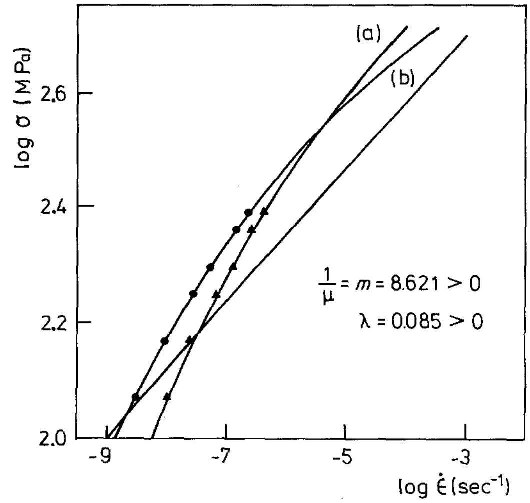

Figure 2 Creep data in cold-worked Zircaloy-4, taken from Fig. 3 of [8] and fitted to Equation 30. $K = 1.93 \times 10^{-31}$ MPa^{-m} sec^{-1}.

3.3. Barrett and Nix theory

Barrett and Nix [7] have proposed a creep model based on the diffusion controlled motion of screw dislocations and the flow law is

$$\dot{\epsilon} = B(\alpha\sigma)^n \sinh(\alpha\sigma) \quad (38)$$

where n is a constant and B and α are parameters that depend on strain for creep, or on the initial stress for stress-relaxation [8]. Equation 38 can be written as

$$y = \log B + n(x + y) + \log \sinh\{\exp[(x + z)/\log e]\} \quad (39)$$

where $x = \log \sigma$, $y = \log \dot{\epsilon}$ and $z = \log \alpha$.

For Equation 39 to be of the form of Equation 4, the following condition must be satisfied

$$\log B = \log C + \beta \log \alpha \quad (40)$$

where C and β are constants. Equation 40 has been already used by Povolo and Marzocca [8, 9]. Introducing Equation 40 into Equation 39 and applying a similar procedure as for Hart's equation, gives for the translation path

$$\Delta \log \dot{\epsilon} = -\beta \Delta \log \sigma \quad (41)$$

The condition of continuity, Equation 17, applied to Equation 39 gives

$$g'(u) = n + \coth\{\exp[(x + z)/\log e]\} \times \exp[(x + z)/\log e] \neq -\beta \quad (42)$$

†Nominal composition (wt %): Sn(1.43), Fe(0.21), Cr(0.1), Zr(balance).

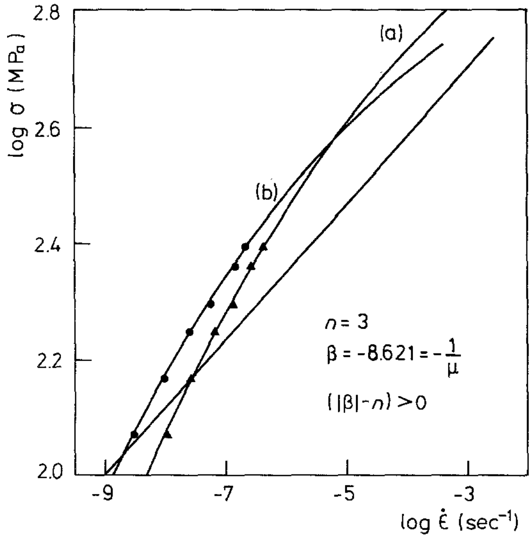

Figure 3 Creep data of Fig. 2 fitted to Equation 39. $C = 3.55 \times 10^{-26}$ MPa$^{\beta}$ sec^{-1}.

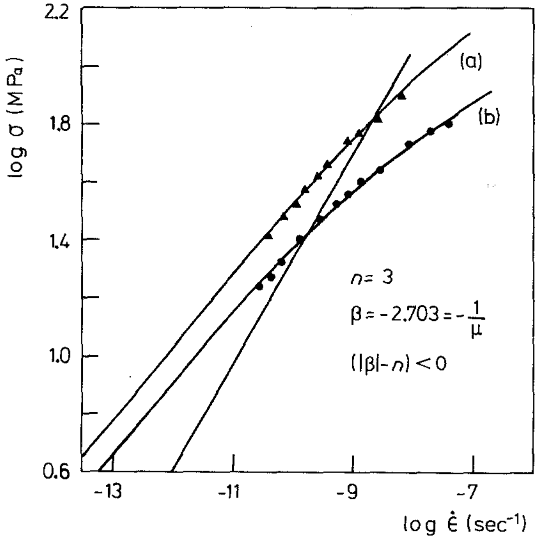

Figure 4 Stress-relaxation data of Fig. 1 fitted to Equation 39. $C = 1.19 \times 10^{-14}$ MPa$^{\beta}$ sec^{-1}.

which shows that there might be crossing points, in the $\log \sigma$–$\log \dot{\epsilon}$ curves, if $(|\beta| - n) > 0$. The crossing point is given by

$$\log \{\sinh \exp [(x + z_2)/\log e] / \sinh \exp [(x + z_1)/\log e]\} = (\beta + n)(z_1 - z_2) \quad (43)$$

As shown by Povolo and Marzocca [8, 9], the creep and stress-relaxation $\log \sigma$–$\log \dot{\epsilon}$ curves for cold-worked Zircaloy-4 can either be fitted to Equation 30 or to Equation 38.

Fig. 3 shows the creep data of Fig. 2 fitted to Equation 39, with the parameters: (a) $\alpha = 10.96 \times 10^{-3}$ (MPa)$^{-1}$, $B = 2.88 \times 10^{-9}$ sec^{-1}; (b) $\alpha = 16.22 \times 10^{-3}$ (MPa)$^{-1}$, $B = 1.32 \times 10^{-10}$ sec^{-1}.

Fig. 4 shows the stress-relaxation data of Fig. 1 fitted to Equation 39, with the parameters: (a) $\alpha = 2.51 \times 10^{-2}$ (MPa)$^{-1}$, $B = 1.95 \times 10^{-10}$ sec^{-1}; (b) $\alpha = 6.53 \times 10^{-2}$ (MPa)$^{-1}$, $B = 1.26 \times 10^{-11}$ sec^{-1}. No crossing point is found for stress-relaxation since $(|\beta| - n) < 0$.

4. Discussion and conclusions

The formalism proposed for the analysis of the existence of an equation of state, in terms of three variables, has proven to be correct. This is shown by the results obtained for the state equation for ideal gases and from the fact that some results, reported previously in the literature, for constitutive equations for plastic behaviour have been obtained as a consequence of the general formalism.

It has been shown that Hart's equation, for low homologous temperatures, can be considered as a state equation. A completely different situation is found, however, for the constitutive equation for high homologous temperatures, even if the $\log \sigma$–$\log \dot{\epsilon}$ curves, for different hardness, are related by scaling. The condition of continuity, in fact, imposes restrictions in the domain of application of Hart's equation for high homologous temperatures. These restrictions are:

(a) $(\sigma^*/\sigma) \geqslant \exp(1/\lambda m)$

(b) $(\sigma^*/\sigma) \leqslant \exp(1/\lambda m)$

for all $\dot{\epsilon}$.

It is not clear which one of the restricted domains should be chosen to describe the plastic behaviour, since the experimental data in cold-worked Zircaloy-4 shows that the creep curves are in domain (a) and the stress-relaxation results in (b). The reciprocal description of these two tests would be impossible, within this model. The condition of continuity, applied to Barrett and Nix model, gives the restricted domains:

(a) $(\alpha\sigma) \coth(\alpha\sigma) \geqslant (-n + |\beta|)$

(b) $(\alpha\sigma) \coth(\alpha\sigma) \leqslant (-n + |\beta|)$

for all $\dot{\epsilon}$. The experimental results in cold-worked Zircaloy-4 do not show a restricted domain for stress-relaxation but do so for creep.

Finally, more creep and stress-relaxation data are needed to clarify the physical meaning of these restriction.

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